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# Star-shaped periodic solutions for $\dot{\mathbf{x}}(t) = -\alpha\{1 - \|\mathbf{x}(t)\|^2\}R(\theta)\mathbf{x}([t])$

大阪電気通信大学工学部　坂田 定久　(Sadahisa Sakata)
大阪府立大学工学部　原　惟行　(Tadayuki Hara)

## 1. Introduction

Recently in [1], Hara considered a 2-dimensional delay differential system

$$\dot{\mathbf{x}}(t) = -\alpha\{1 - \|\mathbf{x}(t)\|^2\}R(\theta)\mathbf{x}(t-1), \tag{1.1}$$

where $\alpha > 0$, $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $|\theta| < \dfrac{\pi}{2}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\|\mathbf{x}\|^2 = x^2 + y^2$. He gave a conjecture :

**Conjecture.** *There exists a constant $\alpha_0 > \pi/2 - |\theta|$ such that $\alpha > \alpha_0$ implies the following* :

(a) *If $\theta/\pi$ is rational, then* (1.1) *has a star-shaped periodic solution.*

(b) *If $\theta/\pi$ is irrational, then each solution orbit densely fills out an annular region centered at the origin.*

Our purpose is to give an answer in some sense to this conjecture for an approximate system to (1.1)

$$\dot{\mathbf{x}}(t) = -\alpha\{1 - \|\mathbf{x}(t)\|^2\}R(\theta)\mathbf{x}([t]), \tag{1.2}$$

where $[\cdot]$ means the greatest-integer function.

We shall consider the system (1.2) together with the initial condition

$$\mathbf{x}(t_0 + s) = \phi(s) \quad \text{for } s \in [-1, 0], \tag{1.3}$$

where $\phi \in C$, the family of all continuous functions from $[-1, 0]$ into $\mathbf{R}^2$. In what follows, $N$ denotes the minimal integer not less than the initial time $t_0$. Then $N = t_0$ if $t_0 \in \mathbf{Z}$, the set of all integers, and $N = [t_0] + 1$ if $t_0 \notin \mathbf{Z}$. Furthermore, $\mathbf{Q}$ means the set of all rational numbers.

Our results in this paper are similar to ones ([2]) for a linear system

$$\dot{\mathbf{x}}(t) = -\alpha R(\theta)\mathbf{x}([t]) \tag{1.4}$$

which is the first approximate system for (1.2).

## 2. Preliminary propositions

In this section, we give preliminary propositions to prove our theorems.

For each solution $\mathbf{x}(t)$ of (1.2) and each integer $n \geq N$, there exists one and only one $\varphi \in [0, 2\pi)$ such that

$$\mathbf{x}(n) = R(\varphi)\begin{pmatrix} \|\mathbf{x}(n)\| \\ 0 \end{pmatrix}. \tag{2.1}$$

Changing variables

$$\mathbf{u}(t) = R(-(\theta + \varphi))\mathbf{x}(t) \tag{2.2}$$

or

$$\mathbf{x}(t) = R(\theta + \varphi)\mathbf{u}(t),$$

we obtain the following proposition.

**Proposition 2.1.** *Let $\mathbf{x}(t)$ be a solution of (1.2). Then $\mathbf{u}(t)$, determined by (2.1) and (2.2), satisfies for any integer $n \geq N$ :*

(a) $\|\mathbf{u}(t)\| = \|\mathbf{x}(t)\|$ *for* $t \geq n$.

(b) $\mathbf{u}(n) = \|\mathbf{x}(n)\| \cdot \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$.

(c) $\dot{\mathbf{u}}(t) = -\alpha\{1 - \|\mathbf{u}(t)\|^2\}\begin{pmatrix} \|\mathbf{x}(n)\| \\ 0 \end{pmatrix}$ *for* $t \in [n, n+1)$.

This proposition follows by elementary calculation and also shows :

**Proposition 2.2.** *Let $\mathbf{x}(t)$ be a solution of (1.2). Then the following are valid :*

(a) $\mathbf{x}(N) = \mathbf{0}$ *implies* $\mathbf{x}(t) = \mathbf{0}$ *for* $t \geq N$.

(b) $\|\mathbf{x}(t_0)\| = 1$ *implies* $\mathbf{x}(t) = \mathbf{x}(t_0)$ *for* $t \geq t_0$.

(c) $\|\mathbf{x}(t_0)\| < 1$ *implies* $\|\mathbf{x}(t)\| < 1$ *for* $t \geq t_0$.

(d) $\|\mathbf{x}(t_0)\| > 1$ *implies* $\|\mathbf{x}(t)\| > 1$, *whenever* $\mathbf{x}(t)$ *exists.*

**Proof.** We prove only (c) and omit the proof of others. First, suppose $\|\mathbf{x}(t_1)\| = 1$ for some $t_1 \leq N$ and $\|\mathbf{x}(t)\| < 1$ on $[t_0, t_1)$. Using change of variables

$$\mathbf{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = R(-(\theta + \varphi_0))\mathbf{x}(t)$$

with

$$\mathbf{x}([t_0]) = R(\varphi_0)\begin{pmatrix}\|\mathbf{x}([t_0])\| \\ 0\end{pmatrix},$$

we have $v(t) = v(t_0)$ and

$$\dot{u}(t) = -\alpha\|\mathbf{x}([t_0])\|\{1 - u(t)^2 - v(t_0)^2\} \tag{2.3}$$

for $t \in [t_0, t_1)$, where $u(t_0)^2 < 1 - v(t_0)^2$. Since $u = \pm\sqrt{1 - v(t_0)^2}$ are critical points for (2.3), uniqueness of solutions for (2.3) guarantees

$$-\sqrt{1 - v(t_0)^2} < u(t) < \sqrt{1 - v(t_0)^2} \quad \text{on } [t_0, t_1],$$

which implies

$$\|\mathbf{x}(t_1)\| = u(t_1)^2 + v(t_1)^2 < 1.$$

This contradicts the supposition $\|\mathbf{x}(t_1)\| = 1$. Therefore $\mathbf{x}(t)$ satisfies $\|\mathbf{x}(t)\| < 1$ on $[t_0, N]$. Next, suppose $\|x(t_1)\| = 1$ for some $t_1 > N$ and $\|\mathbf{x}(t)\| < 1$ on $[t_0, t_1)$. Then there is an integer $n \geq N$ fulfilling $n < t_1 \leq n + 1$. For convenience sake, put

$$\rho = \|\mathbf{x}(n)\|, \quad \beta = \sqrt{1 - \rho^2 \sin^2\theta}.$$

It follows from Proposition 2.1 that

$$\dot{u}(t) = -\alpha\rho\{\beta^2 - u(t)^2\} \tag{2.4}$$

and

$$\dot{v}(t) = 0 \quad \text{or} \quad v(t) = -\rho\sin\theta \tag{2.5}$$

for $t \in [n, n+1)$, where $\mathbf{u}(t) = \begin{pmatrix}u(t) \\ v(t)\end{pmatrix}$. Therefore we can easily show that the inequality $\|\mathbf{x}(t_1)\| < 1$ holds, a contradiction. Thus we conclude that $\|\mathbf{x}(t)\| < 1$ for $t \in [t_0, \infty)$. This completes the proof. □

**Remark 2.1.** *Propositions 2.1 and 2.2 show that every solution* $\mathbf{x}(t)$ *of* (1.2) *with* $\mathbf{x}(N) \neq \mathbf{0}$ *moves straightly from* $\mathbf{x}(n)$ *to* $\mathbf{x}(n+1)$ *as* $t$ *does from* $n$ *to* $n+1$. *Therefore, if* $\|\mathbf{x}(n)\| \to 0$ *as* $n \to \infty$, *then the solution* $\mathbf{x}(t)$ *approaches the origin as* $t \to \infty$. *Furthermore, if* $\mathbf{x}(N+m) = \mathbf{x}(N)$ *for some integer* $m$, *then* $\mathbf{x}(t)$ *runs on a star-shaped periodic orbit for all time.*

Now, we prepare several lemmas for proving our theorems in the next section. Let $0 < \rho < 1$ and put $\beta = \sqrt{1 - \rho^2\sin^2\theta}$. Then it is easy to see $0 < \rho\cos\theta < \beta \leq 1$. So, defining the function $f$ on $(0, 1)$ by

$$f(\rho) = \frac{1}{\rho\beta}\log\frac{\beta + \rho\cos\theta}{\beta - \rho\cos\theta},$$

we obtain the following lemma.

**Lemma 2.1.** *The function $f$ is continuous and strictly increasing in $\rho$, and satisfies*

$$\lim_{\rho\to+0} f(\rho) = 2\cos\theta, \qquad \lim_{\rho\to1-0} f(\rho) = \infty.$$

**Proof.** It is convenient to put

$$g(\rho) = \log\frac{\beta+\rho\cos\theta}{\beta-\rho\cos\theta}.$$

Then it is obvious that $g$ is positive and continuous, and hence $f$ is also. Since $\beta$ tends to 1 as $\rho\to+0$, it follows that $g(\rho)$ tends to 0 as $\rho\to+0$, and so L'Hospital's theorem asserts

$$\lim_{\rho\to+0} f(\rho) = \lim_{\rho\to+0}\frac{g(\rho)}{\rho} = \lim_{\rho\to+0} g'(\rho).$$

Here, elementary calculation shows

$$g'(\rho) = \frac{2\cos\theta}{(1-\rho^2)\beta}.$$

This implies that $f(\rho)$ tends to $2\cos\theta$ as $\rho\to+0$. On the other hand, since $\beta$ tends to $\cos\theta$ as $\rho\to1-0$, the equalities

$$\lim_{\rho\to1-0} f(\rho) = \frac{1}{\cos\theta}\lim_{\rho\to1-0} g(\rho) = \infty$$

hold. Differentiating $f(\rho)$, we have

$$f'(\rho) = \frac{(2\rho\cos\theta)/(1-\rho^2) - g(\rho)\beta + (g(\rho)\rho\sin^2\theta)/\beta}{(\rho\beta)^2}$$

and then

$$f'(\rho) \geq \frac{h(\rho)-g(\rho)}{(\rho\beta)^2}, \tag{2.6}$$

where $h(\rho) = (2\rho\cos\theta)/(1-\rho^2)$. It is easy to see that $h(\rho)$ tends to 0 as $\rho\to+0$ and

$$h'(\rho) = \frac{2\cos\theta(1+\rho^2)}{(1-\rho^2)^2}.$$

Since $1-\rho^2 < \beta^2 \leq \beta < \beta(1+\rho^2)$, it follows that

$$g'(\rho) < \frac{2\cos\theta(1+\rho^2)}{(1-\rho^2)^2} = h'(\rho).$$

This, together with the fact

$$\lim_{\rho\to+0} g(\rho) = \lim_{\rho\to+0} h(\rho) = 0,$$

implies that

$$g(\rho) < h(\rho) \quad \text{for } 0<\rho<1.$$

Hence we can conclude from (2.6) that $f(\rho)$ is strictly increasing in $\rho$. Thus the proof is now completed. □

**Proposition 2.3.** *Let* $\mathbf{x}(t)$ *be a solution of* (1.2) *satisfying* $0 < \|\mathbf{x}(N)\| < 1$. *Then, for any integer* $n \geq N$, *the following are valid* :

(a) $\alpha = f(\|\mathbf{x}(n)\|)$ *implies* $\|\mathbf{x}(n+1)\| = \|\mathbf{x}(n)\|$.

(b) $\alpha < f(\|\mathbf{x}(n)\|)$ *implies* $\|\mathbf{x}(n+1)\| < \|\mathbf{x}(n)\|$.

(c) $\alpha > f(\|\mathbf{x}(n)\|)$ *implies* $\|\mathbf{x}(n+1)\| > \|\mathbf{x}(n)\|$.

**Proof.** In the same manner as the proof of Proposition 2.2, we get (2.4), (2.5),

$$u(n) = \rho\cos\theta \tag{2.7}$$

and also $-\beta < u(t) < \beta$ for $t \in [n, n+1]$. Applying the quadrature to (2.4), we have

$$\frac{\beta + u(n+1)}{\beta - u(n+1)} = \frac{\beta + u(n)}{\beta - u(n)} e^{-2\alpha\rho\beta}. \tag{2.8}$$

On the other hand, $u(n+1) = -u(n)$ if and only if

$$\frac{\beta - u(n+1)}{\beta + u(n+1)} = \frac{\beta + u(n)}{\beta - u(n)}. \tag{2.9}$$

Here, if $\alpha = f(\rho)$, then (2.7) asserts

$$\alpha = \frac{1}{\rho\beta} \log \frac{\beta + u(n)}{\beta - u(n)},$$

and so (2.9) follows from (2.8). Hence we can conclude from Proposition 2.1 (a) and (2.5) that

$$\alpha = f(\|\mathbf{x}(n)\|) \quad \text{implies} \quad \|\mathbf{x}(n+1)\| = \|\mathbf{x}(n)\|.$$

In the same way, we arrive at the conclusion that (b) and (c) of this lemma are valid. □

The following lemma is an immediate consequence of Lemma 2.2 in [2].

**Lemma 2.2.** *There exists a positive integer* $m$ *such that* $R(m(2\theta - \pi)) = I$ *if and only if the ratio* $\theta/\pi$ *is rational.*

## 3. Theorems

Let $\phi$ be an initial function with $\|\phi(0)\| < 1$. Then Proposition 2.2 asserts that the solution $\mathbf{x}(t)$ of (1.2) and (1.3), satisfies $\|\mathbf{x}(t)\| < 1$ on $[t_0, \infty)$. First of all, we give a sufficient condition for such a solution to approach the origin as $t \to \infty$.

**Theorem 3.1.** *Assume* $\alpha \leq 2\cos\theta$. *Then each solution* $\mathbf{x}(t)$ *of* (1.2) *with* $\|\mathbf{x}(t_0)\| < 1$ *approaches the origin as* $t \to \infty$, *and also the zero solution is stable.*

**Proof.** We may assume that $0 < \|\mathbf{x}(N)\| < 1$. Then Lemma 2.1 asserts $\alpha < f(\|\mathbf{x}(N)\|)$. It follows from Proposition 2.3 that

$$\|\mathbf{x}(N+1)\| < \|\mathbf{x}(N)\| < 1.$$

Repeating this argument, we have

$$\|\mathbf{x}(n+1)\| < \|\mathbf{x}(n)\| < 1$$

for any integer $n \geq N$. So, suppose the sequence $\{\|\mathbf{x}(n)\|\}$ converges to a positive $\rho_0$ as $n \to \infty$. Then it is clear that

$$\|\mathbf{x}(n)\| \geq \rho_0 \tag{3.1}$$

for any $n$. Now, consider a system

$$\dot{\mathbf{y}}(t) = -\alpha\{1 - \|\mathbf{y}(t)\|^2\}R(\theta)\boldsymbol{\xi}, \;\; \mathbf{y}(0) = \boldsymbol{\xi}, \tag{3.2}$$

where $\|\boldsymbol{\xi}\| = \rho_0$. Proposition 2.3 asserts that the solution $\mathbf{y}(t;0,\boldsymbol{\xi})$ of (3.2) satisfies

$$\|\mathbf{y}(1;0,\boldsymbol{\xi})\| < \|\boldsymbol{\xi}\| = \rho_0,$$

because $\alpha < f(\|\boldsymbol{\xi}\|)$. Since the set $S = \{\boldsymbol{\xi} \in \mathbf{R}^2 : \|\boldsymbol{\xi}\| = \rho_0\}$ is compact, continuous dependence of solutions on their initial values shows

$$\sup\{\|\mathbf{y}(1;0,\boldsymbol{\xi})\| : \boldsymbol{\xi} \in S\} < \rho_0.$$

Hence there exist a positive $\varepsilon$ and an integer $K$ such that $n \geq K$ implies

$$\|\mathbf{x}(n+1)\| < \rho_0 - \varepsilon,$$

because $\|\mathbf{x}(n)\| \to \rho_0$ as $n \to \infty$. This contradicts (3.1). Therefore we arrive at $\rho_0 = 0$, and so $\|\mathbf{x}(n)\|$ tends to 0 as $n \to \infty$. Thus we conclude from Remark 2.1 that $\mathbf{x}(t)$ approaches the origin as $t \to \infty$. Next, we choose $\boldsymbol{\phi}$ so that

$$(1+\alpha)\|\boldsymbol{\phi}\| < 1,$$

where $\|\boldsymbol{\phi}\| = \sup\{\|\boldsymbol{\phi}(s)\| : -1 \leq s \leq 0\}$. Then it follows from Proposition 2.2 that

$$\|\mathbf{x}(t)\| < 1$$

on $[t_0, \infty)$. Hence (1.2) implies that

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_0)\| + \alpha(t - t_0)\{1 - \|\mathbf{x}(t)\|^2\}\|\mathbf{x}([t_0])\| < (1+\alpha)\|\boldsymbol{\phi}\|$$

for $t \in [t_0, N]$. In particular,

$$\|\mathbf{x}(N)\| < (1+\alpha)\|\boldsymbol{\phi}\|.$$

Since the sequence $\{\|\mathbf{x}(n)\|\}$ is strictly decreasing in $n$, we have from Remark 2.1 that

$$\|\mathbf{x}(t)\| < \|\mathbf{x}(N)\| < (1+\alpha)\|\boldsymbol{\phi}\|$$

for $t \geq N$, and hence

$$\|\mathbf{x}(t)\| < (1+\alpha)\|\boldsymbol{\phi}\|$$

for $t \geq t_0$. This shows that the zero solution is stable. Thus the proof is completed. □

Next, we shall give a sufficient condition for (1.2) to possess star-shaped periodic solutions. This result is a consequence of the following proposition.

**Proposition 3.1.** *Assume that* $\alpha > 2\cos\theta$ *and* $\theta/\pi \in \mathbf{Q}$, *and let* $\mathbf{x}(t)$ *be a solution of* (1.2) *satisfying* $f(\|\mathbf{x}(N)\|) = \alpha$. *Then there exists a positive integer* $m$ *such that*

$$\mathbf{x}(t+m) = \mathbf{x}(t) \tag{3.3}$$

*for* $t \geq N$.

**Proof.** Since domain of $f$ is the interval $(0,1)$, it follows that $0 < \|\mathbf{x}(N)\| < 1$. Then Proposition 2.3 and its proof show

$$\mathbf{u}(N+1) = \|\mathbf{x}(N)\| \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = R(2\theta - \pi)\mathbf{u}(N)$$

or

$$\mathbf{x}(N+1) = R(2\theta - \pi)\mathbf{x}(N),$$

and of course $f(\|\mathbf{x}(N+1)\|) = \alpha$. Repeating this argument, we have

$$\mathbf{x}(N+n) = R(n(2\theta - \pi))\mathbf{x}(N)$$

for any positive integer $n$. Hence Lemma 2.2 ensures the existence of a positive integer $m$ such that

$$\mathbf{x}(N+m) = \mathbf{x}(N). \tag{3.4}$$

Since the system (1.2) is autonomous, we then arrive at the conclusion that

$$\mathbf{x}(t+m) = \mathbf{x}(t)$$

for $t \geq N$. This completes the proof. □

**Theorem 3.2.** *Assume that* $\alpha > 2\cos\theta$ *and* $\theta/\pi \in \mathbf{Q}$. *Then there exist star-shaped periodic solutions of* (1.2).

**Proof.** It follows from Lemma 2.1 that there exists one and only one $\rho \in (0,1)$ satisfying $\alpha = f(\rho)$. Put $\sigma = t_0 - [t_0]$, and choose $\boldsymbol{\phi} \in C$ so that

$$\boldsymbol{\phi}(s) = \begin{pmatrix} \beta \cdot \dfrac{\beta + \rho\cos\theta - (\beta - \rho\cos\theta)e^{2\alpha\rho\beta(s+1)}}{\beta + \rho\cos\theta + (\beta - \rho\cos\theta)e^{2\alpha\rho\beta(s+1)}} \\ -\rho\sin\theta \end{pmatrix}$$

or

$$\frac{\beta - \phi_u(s)}{\beta + \phi_u(s)} = \frac{\beta - \rho\cos\theta}{\beta + \rho\cos\theta} e^{2\alpha\rho\beta(s+1)}, \quad \phi_v(s) = -\rho\sin\theta$$

for $s \in [-1, 0]$, where $\boldsymbol{\phi}(s) = \begin{pmatrix} \phi_u(s) \\ \phi_v(s) \end{pmatrix}$ and $\beta = \sqrt{1 - \rho^2 \sin^2\theta}$. Then it is easy to see that $\phi_u(-1) = \rho\cos\theta$, $\phi_u(0) = -\rho\cos\theta$ and

$$\dot{\phi}_u(s) = -\alpha\rho\{\beta^2 - \phi_u(s)^2\},$$

which implies

$$\dot{\boldsymbol{\phi}}(s) = -\alpha\{1 - \|\boldsymbol{\phi}(s)\|^2\} \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

for $s \in [-1, 0)$. So, define $\boldsymbol{\psi} \in C$ by

$$\boldsymbol{\psi}(s) = \begin{cases} \boldsymbol{\phi}(s + \sigma - 1) & , \ -\sigma \le s \le 0 \\ R(\pi - 2\theta)\boldsymbol{\phi}(s + \sigma), & -1 \le s \le -\sigma. \end{cases}$$

Then the function $\boldsymbol{\psi}$ fulfills

$$\dot{\boldsymbol{\psi}}(s) = -\alpha\{1 - \|\boldsymbol{\psi}(s)\|^2\} \begin{pmatrix} \rho \\ 0 \end{pmatrix} \tag{3.5}$$

for $s \in [-\sigma, 0)$, and

$$\dot{\boldsymbol{\psi}}(s) = -\alpha\{1 - \|\boldsymbol{\psi}(s)\|^2\} R(\pi - 2\theta) \begin{pmatrix} \rho \\ 0 \end{pmatrix} \tag{3.6}$$

for $s \in [-1, -\sigma)$. Now, let $\mathbf{x}(t)$ be the solution of (1.2) with the initial condition

$$\mathbf{x}(t_0 + s) = R(\theta)\boldsymbol{\psi}(s) \quad \text{on} \quad [-1, 0]. \tag{3.7}$$

And, consider the case of $t_0 \notin \mathbf{Z}$. Then $[t_0] = N - 1 < t_0 < N$ and it follows from (3.4) that

$$R(\pi - 2\theta)\mathbf{x}(N + m) = R(\pi - 2\theta)\mathbf{x}(N)$$

and so

$$\mathbf{x}([t_0] + m) = \mathbf{x}([t_0]).$$

Thus $\mathbf{x}(t)$ fulfills

$$\dot{\mathbf{x}}(t) = -\alpha\{1 - \|\mathbf{x}(t)\|^2\} R(\theta)\mathbf{x}([t_0]) \tag{3.8}$$

for $[t_0] + m \le t < N + m$. Furthermore (3.7) implies

$$\mathbf{x}([t_0]) = R(\theta)\boldsymbol{\psi}(-\sigma) = R(\theta)\boldsymbol{\phi}(-1) = \begin{pmatrix} \rho \\ 0 \end{pmatrix}.$$

On the other hand, by (3.7), the equality (3.5) becomes (3.8) for $[t_0] \le t < t_0$. Hence $\mathbf{x}(t)$ fulfills (3.8) on $[[t_0], N)$. By uniqueness of solutions for (3.8), we can conclude that (3.3) holds on $[[t_0], N]$. Similarly, it follows from (3.6) and the equality

$$\mathbf{x}([t_0] - 1 + m) = R(\pi - 2\theta)\mathbf{x}([t_0])$$

that (3.3) holds on $[t_0 - 1, [t_0]]$. Therefore, by Proposition 3.1, we arrive at the conclusion that (3.3) holds for all $t \ge t_0 - 1$. Next, consider the case of $t_0 \in \mathbf{Z}$. Since $t_0 = N$, (3.4) implies

$$\mathbf{x}(t_0 - 1 + m) = R(\pi - 2\theta)\mathbf{x}(t_0 + m) = R(\pi - 2\theta)\mathbf{x}(t_0).$$

On the other hand, (3.6) becomes

$$\dot{\mathbf{x}}(t) = -\alpha\{1 - \|\mathbf{x}(t)\|^2\}R(\theta)R(\pi - 2\theta)\mathbf{x}(t_0).$$

By uniqueness of solutions, we arrive again at the conclusion that (3.3) holds for all $t \ge t_0 - 1$. This shows that $\mathbf{x}(t)$ is a priodic solution, more precisely a star-shaped periodic solution. Moreover, for any $\varphi \in (0, 2\pi)$, the solution of (1.2) with the initial condition

$$\mathbf{x}(t_0 + s) = R(\theta + \varphi)\boldsymbol{\psi}(s) \quad \text{on } [-1, 0]$$

is also periodic. Thus the proof is now completed. □

In the case that $\alpha > 2\cos\theta$ and $\theta/\pi$ is irrational, the system (1.2) does not possess nontrivial periodic solutions. But we obtain a similar result to Theorem 3.4 in [2].

**Theorem 3.3.** *Assume that $\alpha > 2\cos\theta$ and $\theta/\pi \notin \mathbf{Q}$, and let $\mathbf{x}(t)$ be a solution of (1.2) with $f(\|\mathbf{x}(N)\|) = \alpha$. Then the trajectory of $\mathbf{x}(t)$ for $t \ge N$ is everywhere dense on the closed annular region $\{\boldsymbol{\xi} \in \mathbf{R}^2 : \|\mathbf{x}(N)\| \cdot |\sin\theta| \le \|\boldsymbol{\xi}\| \le \|\mathbf{x}(N)\|\}$.*

The proof of this theorem is analogous to one of Theorem 3.4 in [2], and so it is omitted.

Finally we describe a result which is more precise than Proposition 2.2 (d).

**Theorem 3.4.** *Any solution $\mathbf{x}(t)$ of (1.2) with $\|\mathbf{x}(t_0)\| > 1$ possesses a finite escape time $T$, that is, $\|\mathbf{x}(t)\| \to \infty$ as $t \to T - 0$.*

**Proof.** Suppose $\mathbf{x}(t)$ exists in the future. Then it follows from Proposition 2.2 (d) that $\|\mathbf{u}(t)\| > 1$ and so

$$\dot{u}(t) > 0 \quad \text{on } [n, n+1)$$

for each $n \geq N$, where $\mathbf{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is the function determined by (2.1) and (2.2). Since $u(n) > 0$, $\|\mathbf{u}(t)\|$ is strictly increasing in $t$ and hence

$$\|\mathbf{u}(t)\| \geq \rho_N \quad \text{on } [n, n+1),$$

where $\rho_N = \|\mathbf{x}(N)\|$. This implies

$$\dot{u}(t) \geq \alpha\rho_N(\rho_N^2 - 1),$$

so that

$$u(t) \geq \alpha\rho_N(\rho_N^2 - 1)(n - N)$$

on $[n, n+1]$ for each $n \geq N$. Thus we conclude that

$$\|\mathbf{x}(t)\| \to \infty \quad \text{as } t \to \infty. \tag{3.9}$$

Now, consider the case of $\theta \neq 0$. Then there exists a posotive $\rho^*$ such that $\rho > \rho^*$ implies

$$\alpha\rho(\rho^2 \sin^2\theta - 1) > \pi. \tag{3.10}$$

On the other hand, according to the quadrature, we have from Proposition 2.1 (c) that

$$\tan^{-1}\frac{u(t)}{\delta_n} = \tan^{-1}\frac{\rho_n \cos\theta}{\delta_n} + \alpha\rho_n\delta_n(t-n) > \alpha\rho_n\delta_n(t-n)$$

for $t \in [n, n+1)$, where $\rho_n = \|\mathbf{x}(n)\|$ and $\delta_n = \rho_n^2 \sin^2\theta - 1$. But (3.9) implies that $\rho_n > \rho^*$ for $n$ large enough. Hence (3.10) shows that for such an integer $n$, the inequality

$$\tan^{-1}\frac{u(n+\frac{1}{2})}{\delta_n} > \frac{\pi}{2}$$

holds, which is a contradiction. Therefore our supposition is false in the case of $\theta \neq 0$. Next, consider the case of $\theta = 0$. In this case, (c) in Proposition 2.1 becomes

$$\dot{u}(t) = \alpha\rho_n\{u(t)^2 - 1\}, \quad \dot{v}(t) = 0.$$

According to the quadrature again, we have

$$\frac{u(t) - 1}{u(t) + 1} = \frac{\rho_n - 1}{\rho_n + 1} e^{2\alpha\rho_n(t-n)} \tag{3.11}$$

on each interval $[n, n+1)$, because $u(t) > 1$ on $[n, n+1)$. But (3.9) implies that the inequality

$$\frac{\rho_n - 1}{\rho_n + 1} e^{\alpha\rho_n} > 1$$

holds for $n$ large enough. Hence it follows from (3.11) that

$$u\left(n+\frac{1}{2}\right)-1>u\left(n+\frac{1}{2}\right)+1$$

for $n$ above, which is a contradiction. Therefore the solution possesses a finite escape time. This completes the proof. □

## 4. Numerical examples

The following figures are some orbits of (2.1) which illustrate Theorems 3.1 – 3.3.

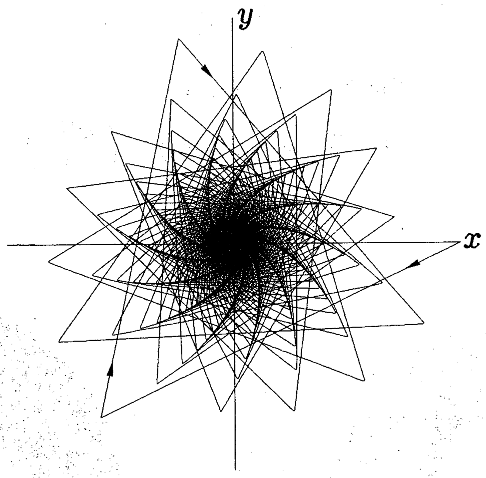


Fig. 1. $\alpha = 1.800 < 2\cos\theta$

$\theta = \frac{\pi}{7},\ t_0 = 0,\ \phi(t) = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$

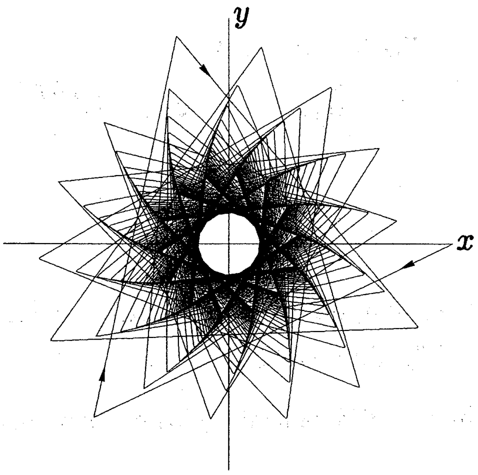


Fig. 2A. $\alpha = 1.805 > 2\cos\theta$

$\theta = \frac{\pi}{7},\ t_0 = 0,\ \phi(t) = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$

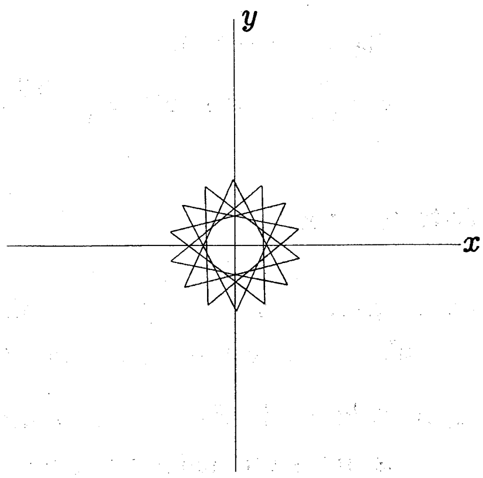


Fig. 2B. $\alpha = 1.805 > 2\cos\theta$

$\theta = \frac{\pi}{7},\ t_0 = 0,\ \phi(t) = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}\ ;\ t \geq 600$

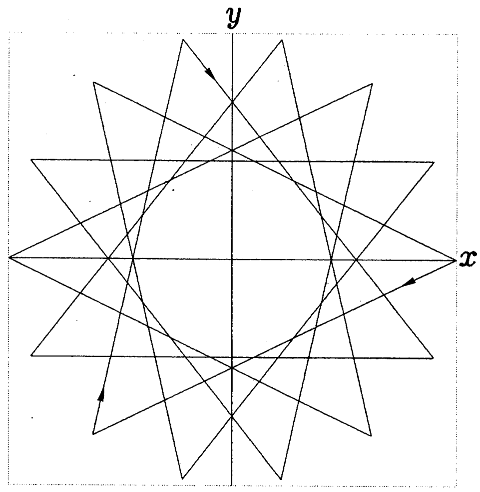


Fig. 3. $\alpha = 8.211$

$$\theta = \frac{\pi}{7}, \quad t_0 = 0, \quad \phi(t) = \begin{pmatrix} 0.999 \\ 0 \end{pmatrix}$$

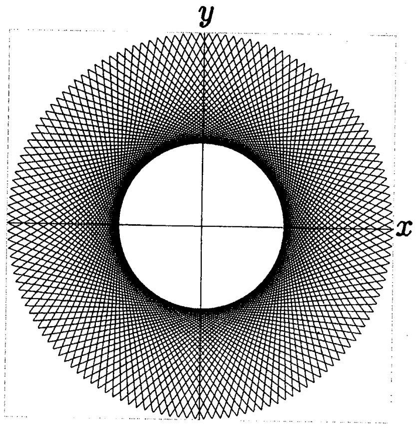


Fig. 4. $\alpha = 8.193$

$$\theta = \frac{\pi}{7.1}, \quad t_0 = 0, \quad \phi(t) = \begin{pmatrix} 0.999 \\ 0 \end{pmatrix}$$

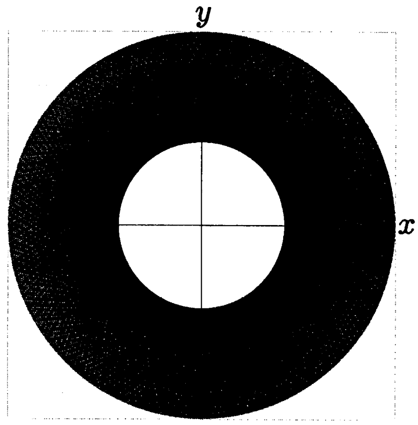


Fig. 5. $\alpha = 8.198$

$$\theta = \frac{\pi}{\sqrt{50}}, \quad t_0 = 0, \quad \phi(t) = \begin{pmatrix} 0.999 \\ 0 \end{pmatrix}$$

# References


[1] T. Hara, The asymptotic stability and star shaped periodic solutions for delay differential system, Nonlinear Anal. 30 (1997) 4555–4563.

[2] S. Sakata, T. Hara, A two-dimensional linear differential system with piecewise constant argument, to appear.